PRECISION OF MEASUREMENTS APPLIED TO PSYCHOMETRIC FUNCTIONS

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In three closely related articles,[1] Dr. F. M. Urban has treated Psychometric Functions by the aid of statistical methods. For convenience these articles will be referred to as *I*, *II*, *III*, respectively. The object of this paper is to discuss his use of these methods from the standpoint of physical measurements only, and not to enter into the psychological questions involved.

In considering the results of physical measurements it is necessary to keep in view the precision of the observations, the extent to which deductions may be carried, and the methods of computation which will give the results without unnecessary labor. It is of course useless to expect good results from insufficient data, and equally useless to sacrifice good data by an incomplete analysis.

In order to be able to discuss the three articles intelligently it is desirable to refer to several fundamental principles of computation.

In most cases the term "precision" should be restricted to fractional precision, *i. e.*, the ratio of the error of a quantity to the entire quantity. When a result of a measurement of any kind is stated as 25,306, it is understood to show that the result lies between 25,305.5 and 25,306.5, or that the value is known with an error of not over one unit in the last digit. The fractional precision is this case is thus, approximately, one part in 25,000. Had the original result been 2,530,600. the fractional precision would have been the same. The two ciphers at the end are "insignificant" digits, serving only to indicate the position of the decimal point. The cipher between 3 and 6 is a significant digit, and the same is true of the ciphers in such a result as .12500. If the last two digits are

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[1]I. The Application of Statistical Methods to the Problems of Psychophysics. The Psychological Clinic Press, Philadelphia, Pa., 1908.

II. Die psychophysischen Massmethoden als Grundlagen empirischer Messungen, Archiv f. d. ges. Psychologie, Vol. 15, Part 3 and 4, Leipzig, 1909.

III. Die psychophysischen Massmethoden als Grundlagen empirischer Messungen, Archiv f. d. ges. Psychologie, Vol. 16, Part 1 and 2, Leipzig, 1909.

written, they indicate, as before, that the result is between .124995 and .125005. When the result lies between .1245 and .1255 the correct form is .125. But in .0012500 the first ciphers are now "insignificant," serving as before to locate the decimal point. When a measurement is actually a counting of individuals, the last digit is not subject to an error in the sense used above. In general, the last digit of a measurement is liable to an error of one half-unit in that place, and to an average error of a quarter-unit. The precision of a measurement may be indicated for practical purposes by stating the number of significant digits, and it is not influenced by the position of the decimal point.

In the case of the four arithmetical processes the precision of the result is usually the same as that of the least precise element, a principle which may be deduced as follows. The product of .234 and 126.5 is 29.6010; but if the given numbers are measurements, they should be considered as .234x and 126.5x, where x indicates unknown digits. In adding the several partial products, each column containing an x must be rejected, leaving the result 29.6. This result is not as impressive as the former, but it is the only one justified by the data. The proof of the rule for division is similar. When a child is taught to annex ciphers to a dividend to facilitate division, he learns a rule which has no place in computation of physical measurements; and a division which is carried on after all the digits of the original dividend have been "brought down" presents a familiar exhibition of false accuracy. In addition and subtraction, the position of the decimal point affects the precision of the result, which is usually the precision of the numerically largest element, the use of an x at the end of each measured element affording a quick and reliable means of testing a result. When two elements in a subtraction are nearly equal, the result is often disappointingly inaccurate, so that an original precision of seven digits may be reduced even to one digit. Of course, logarithmic work is subject to similar criteria. The logarithm of 54.32 taken from a seven-place table may range from 1.7349198 to 1.7349997, so that 1.7349 or 1.7350 is as accurate as the given number will permit. If the given number were 54.32000, indicating a precision of seven digits, the seven place table would be properly chosen for use. Conversely, if the logarithm of a number is 2.34127, the number is anywhere from 219.4143 to 219.4194, *i. e.*, is correct to five digits only. Thus, in general, the number of places in the log. table should be that of the digits in the number. With few exceptions the precision of a result is not more than that of the data, and it is usually less.

It will be necessary later to employ the term average deviation of the mean. If the sum of n quantities is divided by n, the result is the arithmetic mean. The sum of the differences between the arithmetic mean and the quantities, taken without regard to sign, and divided by n, is the average deviation of a single observed quantity. If this deviation is divided by $\sqrt{n}$, the result is called the average deviation of the mean and gives a precision measure in common use by physicists. It is customary to compute this to two places of significant figures and then to retain in the arithmetic mean no digit beyond this second digit, since more than these are useless. The combined result is often written in the form $35.123 \pm .012$.

The experiments which were the basis of Dr. Urban's articles are described in *I*, page 1, and in *II*, page 261. A set of brass cylinders externally identical and of weights varying by four grams from 84 to 104 gms. was arranged at equidistant intervals determined by numbers from 1 to 14, around the circumference of a circular table. Standard weights of 100 gms. were placed at the odd numbers. The individual to be tested was requested to lift each weight in turn, and to give his judgment as to the relative weight of each cylinder at an even number and the standard cylinders at the preceding odd number. The table was rotated so as to bring the weights in succession under the hand of the observer, who stated his judgments at the rate of eleven and one-half per minute. Each experiment consisted of 50 comparisons of each weight with the standard, and the results of each experiment were separately tabulated, the judgments being classified as heavier, lighter or equal.

Seven observers were employed, of whom the first three performed nine experiments, and the others six. The "frequencies" for each of the three types of judgment were computed by dividing the total number of judgments of each type and for each weight by the total number of judgments in the nine or six experiments. Observer I° gave the judgment "equal" 28 times for the 84 gm. weight and 56 times for the 88 gm. weight. So that the two frequencies were $28 \div 450$ and $56 \div 450$ respectively. This process gave eventually a table of frequencies having seven entries, one for each weight. By interpolation, frequencies were found corresponding to weights varying by single grams from 84 to 108, but only the seven original entries were experimental results. After the plotting of these extended results for each observer and for each type of judgment, smooth curves were drawn through the twenty-five points of each plot. The curves for equality judgments were somewhat like the ordinary probability curve, while those

for lighter and heavier judgments were low and high respectively at the right ends and *vice versa* at the left ends.

The equality curve for observer I° was treated most elaborately and so will require the most attention in this paper. The original frequencies for observer I° (*I*, table 85) were:

.0622, .1244, .3311, .4422, .4644, .0911, .0533.

The number 28 above mentioned, which is the total for nine experiments, is the sum of widely varying components, viz., 4, 3, 4, 4, 2, 5, 2, 3, 1. If we follow the procedure previously explained and compute the average deviation of the mean, these values give 3.11 ± .34. In this manner the original frequencies revised and with useless digits omitted are:

.0622 ± .0067, .124 ± .019, .331 ± .045, .442 ± .025.
.464 ± .045, .0911 ± .0061. .0533 ± .0099.

Thus a three place log. table is sufficiently accurate for the computation, though it must not be inferred that this means rough approximation; for, on the scale adopted by Dr. Urban, a change of a unit in the third decimal place of an ordinate is not visible to the naked eye. Granting the use of these ordinates to four digits and no further, it should be observed that these ordinates and the corresponding abscissae 84, 88, etc., give seven points and no more for the purpose of defining the psychometric curve of this observer.

At *I*, page 126, Dr. Urban proceeded by the use of Lagrange's formula to obtain the equation of this curve in Cartesian co-ordinates, since in that form the result is easily differentiated, thus enabling one to locate the maximum ordinate. In this computation the seven ordinates were treated as if exact to any desired extent, *i. e.*, in the divisions, zeros were added to the dividends giving some quotients to twenty-four significant figures. Such precision is equivalent to stating the volume of the earth with an error of not over one cubic inch. On an inserted sheet (*I*, p. 129), giving only a portion of the details, there are over fifteen hundred digits, and but for two accidents the equation of the psychometric curve would have coefficients correct to eighteen digits. One accident is that certain of the results are computed to only eleven digits; the other is an error in the computation of a,ϕ (x): (x—84) m_1, where the coefficient of X^4 is given as .003035422098013889, when it should be .003035422092013889. This error gives the ordinate of the point for which X=100 the value .4650 instead of .4644 as in the data. Newton's method of differences gives a formula which is theoretically the same as that by Lagrange, and requires about one-tenth of the labor. While this is not in a form for easy differentiation, the method of approximation will quickly locate the maximum ordinate, giving its location far closer than the data will warrant.

Certain theoretical topics about this curve must now be treated. In *I*, page 139, this statement occurs. "Interpolation by Lagrange's formula has not the character of a definite hypothesis on the nature of the psychometric function but it is rather a method of completing a set of observations." Lagrange's formula gives the equation of a curve through *n* points, whose degree is not greater than *n* and the *n* points dedetermine the curve completely. In fact, for any set of *n* points only one curve exists of the type which Lagrange's formula assumes. The use of Lagrange's formula certainly makes a definite assumption about the type of the psychometric function, which is all the more open to objection because all of the probability curves, both symmetrical and asymmetrical, are definitely excluded. An infinity of curves of the same general form, if desired, but of higher degree may be made to pass through any *n* points, so that it seems utterly useless to spend energy in locating within one one-thousandth of an inch the maximum point of a curve whose only claim for consideration is the fact that its equation is simpler than that of other curves. Circles were once considered the only perfect curves; hence it was argued that heavenly bodies must have circular orbits. As a last comment on the quotation above, if Lagrange's formula enables one to complete a set of observations, why not make fewer observations and use the formula instead? Thus from two observations one could obtain an indefinite number of new "observations," but unfortunately all would lie on a straight line through the two original points.

Without detracting from the very able mathematical treatment of the theoretical psychometric curves, it is important to notice that the observation curves in *I* and *II* show such divergence from the theoretical curves in *III* that deductions from the latter are not applicable to the former. But the former are not a necessary conclusion from the seven points. From a mathematical standpoint it is desirable to have more points for the curves, and these must not be obtained by any formula of interpolation, since this process introduces an assumption about the curves, in fact is equivalent to defining the curves completely. Under the conditions provided in these experiments the deductions may quite as well be obtained from the plots themselves, since the observations are not sufficiently precise to justify such exhaustive mathematical treatment.